Powers of a Matrix and a Formula for the Moduli of its Eigenvalues

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Dedicated to Helmut Wielandt on his 75th birthday

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ABSTRACT

The spectral radius of a complex square matrix A is given by $\rho(A) = \limsup_{k\to\infty}(\operatorname{Tr} A^k)^{1/k}$. A more general result is proved which gives information about the moduli of all eigenvalues of A.

It is well known that the spectral radius $\rho(A)$ of a complex $n \times n$ matrix A can be described by

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k} \tag{1}$$

where $\|A\| = \sqrt{\rho(AA^*)}$ is the spectral norm of A (see e.g. [1]). The traces of powers [2] yield another limit for $\rho(A)$,

$$\rho(A) = \limsup_{k\to\infty} |\operatorname{Tr} A^k|^{1/k}. \tag{2}$$

Equation (1) is a special case of the following result of Yamamoto [3].

THEOREM 1 [3]. *Let the eigenvalues* λ_i, $i = 1,\dots,n$, *of* $A \in \mathbb{C}^{n\times n}$ *be arranged such that*

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|, \tag{3}$$

and let

$$\alpha_1^{(k)} \geqslant \alpha_2^{(k)} \geqslant \cdots \geqslant \alpha_n^{(k)}$$

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be the singular values of A^k. *Then*

$$|\lambda_i| = \lim_{k\to\infty} \left(\alpha_i^{(k)}\right)^{1/k}, \qquad i=1,\dots,n. \tag{4}$$

The relation between (1) and Theorem 1 suggests that (2) is also a special case of a more general result. It is the purpose of this note to prove the following fact.

THEOREM 2. *Let the eigenvalues* λ_i *of* $A \in \mathbb{C}^{n\times n}$ *be arranged as in* (3), *and let*

$$z^n + \beta_{n-1}^{(k)} z^{n-1} + \cdots + \beta_1^{(k)} z + \beta_0^{(k)}$$

be the characteristic polynomial of A^k. *Then*

$$|\lambda_1\lambda_2\cdots\lambda_i| = \limsup_{k\to\infty} |\beta_{n-i}^{(k)}|^{1/k}, \qquad i=1,\dots,n. \tag{5}$$

Proof. We denote the ith compound matrix of A, which contains all $i\times i$ minors of A, by $C_i(A)$ and recall [1] that the eigenvalues of $C_i(A)$ are the products $\lambda_{m_1}\lambda_{m_2}\cdots\lambda_{m_i}$, $1 \leqslant m_1 < m_2 < \cdots < m_i \leqslant n$. Hence $\rho(C_i(A)) = |\lambda_1\lambda_2\cdots\lambda_i|$. Note that $\beta_{n-i}^{(k)} = \operatorname{Tr} C_i(A^k)$. Because $C_i(A^k) = [C_i(A)]^k$, we deduce from (2)

$$\limsup_{k\to\infty} \left|\operatorname{Tr} C_i(A^k)\right|^{1/k} = \rho(C_i(A)) = |\lambda_1\lambda_2\cdots\lambda_i|. \qquad \blacksquare$$

In the case $i=1$ of (5) we obtain (2) with $|\lambda_1| = \rho(A)$ and $\beta_{n-1}^{(k)} = \operatorname{Tr} A^k$.

I am grateful to J. Zemánek for a valuable discussion.

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Received 4 September 1984; revised 5 November 1984